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INFINITE GAMES AND GENERALIZATIONS OF FIRST-COUNTABLE SPACES

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In this paper we introduce a new class of spaces, called W-spaces, which is defined in terms of a simple two-person infinite game. Every first-countable space is a W-space, and every W-space is countably bi-sequential. The W-space property is preserved by subspaces, Σ-products, and open mappings. Separable W-spaces are first-countable. Various other properties of W-spaces are studied, and some questions are posed.

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W-space	first-countable
Fréchet space	bi-sequential
c-space	countably bi-sequential

1. Introduction

Generalizations of first-countable spaces (e.g., sequential spaces, q-spaces, Fréchet spaces, etc.) have been the object of study by many authors, too numerous to mention here. In this paper we introduce a generalization of first-countable spaces which seems to bear an interesting relationship to a number of others. We call these spaces W-spaces.

W-spaces are defined in terms of a simple two-person infinite game. We also consider an (apparently) slight generalization of W-spaces, called w-spaces, which are actually the same as W-spaces if and only if all such games are determined (i.e., there is a winning strategy for one player or the other). We are particularly interested in the relationship of W-spaces to the following classes of spaces:

(i) first-countable spaces;
(ii) bi-sequential spaces;

(iii) countably bi-sequential spaces;
(iv) Fréchet spaces;
(v) c-spaces, or spaces determined by countable subsets.

The basic relationships among these spaces are indicated in the following diagram:

$$\begin{tikzcd}
& \text{first-countable} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\text{W-space} \arrow[d, Rightarrow] & & \text{bi-sequential} \arrow[dddl, Rightarrow] \\
\begin{array}{c}\text{c-space}\\ + \\ \text{countable subsets}\\ \text{are first-countable}\end{array} \arrow[d, Rightarrow] & & \\
\text{w-space} \arrow[dr, Rightarrow] & & \\
& \text{countably bi-sequential} \arrow[d, Rightarrow] & \\
& \text{Fréchet space} \arrow[d, Rightarrow] & \\
& \text{c-space} &
\end{tikzcd}$$

Note that both the class of bi-sequential spaces and the class of W-spaces lie between first-countable spaces and countably bi-sequential spaces. These two classes have other similarities: both are preserved by subspaces, countable products, and open mappings. However, W-spaces are preserved by Σ-products (bi-sequential spaces are not), while bi-sequential spaces are preserved by bi-quotient images (W-spaces are not). Also, separable W-spaces are first-countable, but separable bi-sequential spaces need not be first-countable.

Michael [7, Proposition 8.7] has shown that a space is countably bi-sequential if and only if it is a c-space and every countable subset is countably bi-sequential. As indicated above, every W-space is a c-space in which every countable subset is first-countable. We do not know if the reverse is true. Indeed, we do not know of a w-space which is not a W-space, and it rather surprised us to find a class of spaces which fits between them. The W-spaces, the strongest of the three, seem to be the easiest to work with. Indeed, W-spaces behave as well as first-countable spaces in many instances, especially with respect to products. Even so, there are many common examples of

W-spaces which are not first-countable, e.g., the one-point compactification of an uncountable discrete space. W-spaces are surely easier to work with than c-spaces in which every countable subset is first-countable. In fact, several theorems in this paper about W-spaces are unknown (to the author, at least) for the latter class of spaces.

Section 2 contains the definitions, and in Section 3 we develop the basic properties of W-spaces, including the implications in the above diagram. In Section 4 we study products of W-spaces with themselves and with the other classes of spaces mentioned above. Section 5 is devoted to examples.

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2. Definitions

(1) Let x be a point in the topological space X, and consider the following two-person infinite game: player I chooses an open set U_1 containing x, and then player II chooses a point $x_1 \in U_1$; player I then chooses another open set U_2 containing x, player II chooses some point $x_2 \in U_2$, and so on. We shall say that player I wins the game if the sequence $\langle x_1, x_2, ...\rangle$ converges to x.

More precisely, a *strategy at x for player* I is a map $\sigma : \mathcal{F}(X) \to \mathcal{T}_x(X)$, where $\mathcal{F}(X)$ is the set of all finite sequences in X, and $\mathcal{T}_x(X)$ is the set of open subsets of X containing x. A *σ-sequence* of the strategy σ is a sequence $\langle x_1, x_2, ...\rangle$ such that $x_{n+1} \in \sigma(\langle x_1, ..., x_n\rangle)$ for all $n \in \mathbf{N}$. [We shall assume $\sigma(\emptyset) = X$. Note that $\sigma(\langle x_1, ..., x_n\rangle)$ represents the open set player I would choose if $\langle x_1, ..., x_n\rangle$ have been the first n choices for player II, and a σ-sequence represents the result of a game that has been played.] Such a strategy σ is called a *winning strategy* at x if every σ-sequence converges to x. We shall call a space X a *W-space* if there exists a winning strategy for player I at each point of X.

(2) A *strategy at x for player* II is a map $\tau : \mathcal{F}(X) \times \mathcal{T}_x(X) \to X$ such that $\tau(F, U) \in U$ for each $F \in \mathcal{F}(X)$ and $U \in \mathcal{T}_x(X)$. We shall call a space X a *w-space* if for every strategy τ for player II there exists a counterstrategy $\sigma(\tau)$ which wins for player I. Note that this does not necessarily imply that every $\sigma(\tau)$-sequence converges. Only that sequence corresponding to the game played with these two strategies need converge. However, as mentioned in the introduction, we do not know of a w-space which is not a W-space. Such a space exists if and only if there is such a game which is not determined.

(3) A space X is said to be *bi-sequential* if whenever $x \in X$ and $\mathcal{F}$ is an ultrafilter in X which clusters at x, then there exists a decreasing sequence $\langle A_1, A_2, \ldots \rangle$ of elements of $\mathcal{F}$ which converges to x (i.e., $A_1 \supset A_2 \supset \ldots$, and if U is an open set containing x, then there is $n_0 \in \mathbf{N}$ such that $A_n \subset U$ for all $n > n_0$).

(4) A space X is said to be *countably bi-sequential* if for every decreasing sequence $\langle F_1, F_2, \ldots \rangle$ such that $x \in \bigcap \{\overline{F}_n \mid n = 1, 2, \ldots\}$, there is $x_n \in F_n$ such that $x_n \to x$.

(5) A space X is called a *Fréchet space* if whenever $x \in \overline{A}$ there are $x_n \in A$ such that $x_n \to x$.

(6) A space X is called a *c-space* if whenever $x \in \overline{A}$ there is a countable subset $C \subset A$ such that $x \in \overline{C}$. (c-spaces have also been called "spaces determined by countable subsets" and "spaces of countable tightness".)

3. Basic properties

Theorem 3.1. *Every subspace of a* W*-space* (w*-sapce*) *is a* W*-space* (w*-space*).

Proof. If $\sigma : \mathcal{F}(X) \to \mathcal{T}_x(X)$ is a winning strategy at $x \in X$, and $x \in X' \subset X$, define $\sigma' : \mathcal{F}(X') \to \mathcal{T}_x(X')$ by $\sigma'(F) = \sigma(F) \cap X'$. Clearly σ' is a winning strategy at $x \in X'$. The proof for w-spaces is similar. □

Theorem 3.2. *Every first-countable space is a* W*-space; every* w*-space is countably bi-sequential.*

Proof. Suppose X is first-countable. Let $x \in X$ and let $\{U_n\}_{n=1}^{\infty}$ be a countable decreasing local basis at x. Clearly $\sigma(\langle x_1, \ldots, x_n \rangle) = U_n$ defines a winning strategy at x for player I.

To prove the second statement, let Y be a w-space and let $\langle F_1, F_2, \ldots \rangle$ be a decreasing sequence such that $y \in \overline{F}_n$ for all n. Let τ be any strategy for player II such that $\tau(\langle y_1, \ldots, y_n \rangle, U) \in F_{n+1}$ for every sequence $\langle y_1, \ldots, y_n \rangle$ of n elements of Y. A counterstrategy $\sigma(\tau)$ then yields a sequence $y_n \in F_n$ such that $y_n \to y$, and the proof is finished. □

Note that the above theorem implies that if a space is not countably bi-sequential, then there must exist a winning strategy for player II.

If $\sigma : \mathcal{F}(X) \to \mathcal{T}_x(X)$ is a strategy at $x \in X$, let $R(\sigma)$ be the range of σ. The character $\chi(x)$ is defined to be the minimum cardinality of a basis at

x, and

$$\chi(X) = \sup\{\chi(x) : x \in X\}.$$

Theorem 3.3. *If σ is a winning strategy at $x \in X$, then $R(\sigma)$ is a basis at x. Furthermore,*

$$\chi(x) = \min\{|R(\sigma)| \mid \sigma \textit{ is a winning strategy at } x\} \leqslant |X|.$$

Proof. Suppose $R(\sigma)$ is not a basis at x. Then there is an open set U containing x such that for each $V \in R(\sigma)$, $V - U \neq \emptyset$. But then if player I uses σ, player II could always choose a point not in U, yielding a σ-sequence which does not converge to x. Thus $R(\sigma)$ must contain a basis at x.

If $|R(\sigma)| > \chi(x)$, let B be a basis at x of cardinality $\chi(x)$, and define a strategy σ' such that for every $F \in \mathcal{F}(X)$ we have $x \in \sigma'(F) \subset \sigma(F)$ and $\sigma'(F) \in B$. Clearly σ' is also a winning strategy at x and $\chi(x) = |R(\sigma')|$. Also, $|R(\sigma')| \leqslant |\mathcal{F}(X)| = |X|$, and the proof is finished. □

Corollary 3.4. *A* W*-space is a* c*-space in which every countable subset is first-countable.*

Proof. That a W-space is a c-space follows from Theorem 3.2, and that every countable subset is first-countable follows from Theorem 3.1 and Theorem 3.3. □

Corollary 3.5. *If X is a* W*-space, then $w(X) \leqslant |X|$.*

(The *weight* $w(X)$ is the minimum cardinality of a basis for X.)

Proof. By Theorem 3.3, $\chi(X) \leqslant |X|$. Thus

$$w(X) \leqslant \chi(X) \cdot |X| \leqslant |X| \cdot |X| = |X|. \ \square$$

By [4, Theorem 3], a regular separable space in which every countable subset is first-countable must itself be first-countable. Thus a regular separable W-space is first-countable. We have the following more general result.

Theorem 3.6. *If X is a regular* W*-space, then $\chi(X) \leqslant d(X)$, where $d(X)$ is the minimum cardinality of a dense subset of X.*

Proof. Let $x \in X$, and let S be a dense subset of X such that $|S| = d(X)$. Let $\sigma : \mathcal{F}(X) \to \mathcal{T}_x(X)$ be a winning strategy at x, and let $\sigma' = \sigma|_{\mathcal{F}(S)}$. Then $|R(\sigma')| \leqslant |S|$. Thus we need only show that $R(\sigma')$ is a basis at x. Suppose not. Then there exists an open set U containing x such that for every $V \in R(\sigma')$ we have $V - \overline{U} \neq \emptyset$. But then if player I uses σ, player II can always choose an element of S not in $\overline{U}$, yielding a σ-sequence which does not converge to x. Thus the theorem is proved. □

The following theorem gives a clearer picture of the local structure of W-spaces.

Theorem 3.7. *Let X be T_2. If there exists a winning strategy σ for player I at $x \in X$, then either x is G_δ in X, or there exists an uncountable discrete subset $D \subset X$ such that $D \cup \{x\}$ is compact.*

Proof. If x is not G_δ in X, then there exists a set $D = \{x_\alpha \mid \alpha < \omega_1\}$ such that for each $\beta < \omega_1$ we have

$$x_\beta \in \bigcap \{\sigma(F) \mid F \text{ is a finite sequence in } \{x_\alpha \mid \alpha < \beta\}\}.$$

It is easy to see that every countable subset of D contains a σ-sequence, hence clusters at x. Thus every open set containing x contains all but finitely many elements of D, and the proof is finished. □

Theorem 3.8. *If X is a c-space in which every countable subset is first-countable, then X is a w-space.*

Proof. Suppose X satisfies the hypothesis of the theorem, but that X is not a w-space. Then there exists a strategy $\tau : \mathcal{F}(X) \times \mathcal{T}_x(X) \to X$ for player II for which there is no counterstrategy for player I. For each $F \in \mathcal{F}(X)$, define

$$\tau_F = \{\tau(F, U) \mid U \in \mathcal{T}_x(X)\}.$$

Observe that for each F, τ_F clusters at x.

Let $c_0 = \tau(\emptyset, X)$. (Recall that we have assumed $\sigma(\emptyset) = X$, so $\tau(\emptyset, U)$ for $U \neq X$ is not the beginning of any game.) Let $C_0 = \{c_1, c_2, ...\}$ be a countable subset of $\tau_{\langle c_0\rangle}$ which clusters at x. For each $n \in \mathbf{N}$, let $C_n = \{c_{n1}, c_{n2}, ...\}$ be a countable subset of $\tau_{\langle c_0, c_n\rangle}$ which clusters at x. If $C_{n_1 \ldots n_j} = \{c_{n_1 \ldots n_j m} \mid m \in \mathbf{N}\}$ has been defined for all $(n_1, ..., n_j) \in \mathbf{N}$,

$j < k$, we let $C_{n_1 \dots n_k} = \{c_{n_1 \dots n_k m} \mid m \in \mathbf{N}\}$ be a countable subset of τ_F which clusters at x, where

$$F = \langle c_0, c_{n_1}, c_{n_1 n_2}, \dots, c_{n_1 \dots n_k} \rangle .$$

Let

$$C = \bigcup \{C_{n_1 \dots n_k} \mid (n_1, \dots, n_k) \in \mathbf{N}^k, k \in \mathbf{N}\} \cup C_0 .$$

We claim that the countable subspace $X' = C \cup \{x\}$ of X is not a W-space, hence not first-countable. To see this, observe that if $\{c_{n_1}, c_{n_1 n_2}, \dots, c_{n_1 \dots n_k}\}$ are the first k choices of player II, player II can always choose some $c_{n_1 \dots n_k n_{k+1}} \in C_{n_1 \dots n_k}$ as his next choice. However, every sequence $\langle c_0, c_{n_1}, c_{n_1 n_2}, \dots \rangle$ corresponds to a game played in X with player II using τ, and thus does not converge to x. Thus X' is not a W-space, and the theorem is proved. □

It may be suggested that we alter the game we are considering by declaring that player I wins at x if he can merely force the points of player II to cluster at x. It is useful to know that this game is equivalent to the usual game in the following sense:

Theroem 3.9. *Let $x \in X$, and suppose σ is a strategy for player I at x such that every σ-sequence clusters at x. Then there exists a winning strategy for player* I *at x.*

Proof. Define $\sigma' : \mathcal{F}(X) \to \mathcal{T}_x(X)$ by

$$\sigma'(F) = \bigcap \{\sigma(G) \mid G \text{ is a subsequence of } F\}.$$

Then if $S = \langle x_1, x_2, \dots \rangle$ is a σ'-sequence, every subsequence of S is a σ-sequence. Thus every subsequence of S clusters at x, which implies that S converges to x.

A map $f : X \to Y$ is *almost open* if for every $y \in Y$ there exists an $x \in f^{-1}(y)$ having a basis of open neighborhoods in X whose images are open in Y.

Theorem 3.10. *The almost open image of a* W*-space is a* W*-space.*

Proof. Let $f : X \to Y$ be almost open, with X a W-space. Let $y \in Y$, and let $x \in f^{-1}(y)$ be as in the definition of almost open. Let σ be a winning strategy for player I at x. We define a strategy σ' for player I at y as follows. If y_1 is the first choice of player II, choose $x_1 \in f^{-1}(y_1)$ and let $\sigma'(\langle y_1\rangle) = f(\sigma(\langle x_1\rangle))$. If $\sigma'(\langle y_1, ..., y_n\rangle)$ has been defined, and y_{n+1} is the next choice of player II, choose $x_{n+1} \in (f^{-1}(y_{n+1})) \cap \sigma(\langle x_1, ..., x_n\rangle)$, and let

$$\sigma'(\langle y_1, ..., y_n\rangle) = f(\sigma(\langle x_1, ..., x_{n+1}\rangle)).$$

Now if $\langle y_1, y_2, ...\rangle$ is a σ'-sequence, then $\langle x_1, x_2, ...\rangle$ is a σ-sequence. Thus $x_n \to x$, which implies $y_n \to y$. □

Every almost open map is bi-quotient. The above theorem is not, however, true for bi-quotient maps, since there are bi-sequential spaces which are not W-spaces.

4. Product theorems

Theorem 4.1. *The countable product of* W*-spaces is a* W*-space.*

Proof. Let $X = \Pi_{n=1}^{\infty} X_n$, where the X_n's are W-spaces. Let $x = (x^1, x^2, ...) \in X$, and let σ_i be a winning strategy for player I at $x^i \in X_i$. If $F \in \mathcal{F}(X_i)$, let

$$\hat{\sigma}_i(F) = \{y \in X \mid y^i \in \sigma_i(F)\}.$$

Define a strategy σ for player I at $x \in X$ by

$$\sigma(\langle y_1, ..., y_n\rangle) = \bigcap_{i=1}^{n} \hat{\sigma}_i(\langle y_1^i, ..., y_n^i\rangle).$$

It is easy to check that σ is a winning strategy. □

The product of two c-spaces need not be a c-space, even if each factor is a Fréchet space [1]; see also Example 5.3. In fact, assuming the continuum hypothesis, Malyhin [6] has an example of a c-space X and a countably bi-sequential space Y such that $X \times Y$ is not a c-space. However, we have the following theorem:

Theorem 4.2. *The product of a* c*-space with a* W*-space is again a* c*-space.*

Proof. Let X be a c-space, Y a W-space, $(x_0, y_0) \in \overline{H} - H$, $H \subset X \times Y$.

Let σ be a winning strategy for player I at $y_0 \in Y$, and for $F \in \mathcal{F}(Y)$ define

$$\sigma(F)_H = \{x \in X \mid (x, y) \in H \text{ for some } y \in \sigma(F)\}.$$

Observe that x_0 is always in the closure of $\sigma(F)_H$. Let $C_0 = \{c_1, c_2, ...\}$ be a countable subset of $\sigma(\emptyset)_H$ such that $x_0 \in \overline{C}_0$. For each $c_n \in C_0$, choose $c'_n \in \sigma(\emptyset)$ such that $(c_n, c'_n) \in H$.

For each $n \in \mathbf{N}$, choose a countable subset

$$C_n = \{c_{n1}, c_{n2}, ...\} \subset \sigma(\langle c'_n \rangle)_H$$

such that $x_0 \in \overline{C}_n$. For each $m \in \mathbf{N}$, choose $c'_{nm} \in \sigma(\langle c'_n \rangle)$ such that $(c_{nm}, c'_{nm}) \in H$. We continue inductively, defining $C_{n_1 ... n_k} = \{c_{n_1 ... n_k m} \mid m \in \mathbf{N}\}$ and $c'_{n_1 ... n_k}$ for all $(n_1, ..., n_k) \in \mathbf{N}^k$, $k \in \mathbf{N}$, with the following properties:

(i) $x_0 \in \overline{C}_{n_1 ... n_k}$;

(ii) $(c_{n_1 ... n_k}, c'_{n_1 ... n_k}) \in H$;

(iii) $c'_{n_1 ... n_{k+1}} \in \sigma(\langle c'_{n_1}, c'_{n_1 n_2}, ..., c'_{n_1 n_2 ... n_k} \rangle)$.

Let

$$C = \bigcup \{C_{n_1 ... n_k} \mid (n_1, ..., n_k) \in \mathbf{N}^k, k \in \mathbf{N}\} \cup C_0.$$

We claim that C is a countable subset of H such that $(x_0, y_0) \in \overline{C}$. To see this, let $(x_0, y_0) \in U \times V$, where U and V are open in X and Y, respectively. Choose $c_{n_1} \in U \cap C_0$. If $c_{n_1 ... n_k}$ has been chosen, choose $c_{n_1 ... n_{k+1}} \in U \cap C_{n_1 ... n_k}$. Since $\langle c'_{n_1}, c'_{n_1 n_2}, ... \rangle$ is a σ-sequence, there exists an integer m such that $c'_{n_1 ... n_m} \in V$. Thus $(c_{n_1 ... n_m}, c'_{n_1 ... n_m}) \in (U \times V) \cap H$, and the proof is finished. □

The product of two countably bi-sequential spaces need not be countably bi-sequential, even if they are compact [2]. As a corollary to Theorem 4.2, we have the following positive result:

Corollary 4.3. *The product of a countably bi-sequential space with a W-space is countably bi-sequential.*

Proof. Let X be countably bi-sequential, and let Y be a W-space. By Theorem 4.2, $X \times Y$ is a c-space. Furthermore, every countable subset of $X \times Y$ is a subspace of the product of a countably bi-sequential space with a first-countable space, and therefore is countably bi-sequential by [7, Proposition 4.D.4]. Then by [7, Proposition 8.7], $X \times Y$ is countably bi-sequential. □

Corollary 4.4. *Let X be a c-space in which every countable subset is first-countable, and let Y be a W-space. Then $X \times Y$ is a c-space in which every countable subset is first-countable.*

Proof. This corollary follows from Theorem 4.2 and the fact that every countable subset of $X \times Y$ is a subset of the product of first-countable spaces. □

We do not know if the product of two w-spaces is again a w-space. However, the following is true.

Theorem 4.5. *The product of a w-space with a W-space is a w-space.*

Proof. Let X be a w-space, Y a W-space, $(x, y) \in X \times Y$, and suppose τ is a strategy for player II at (x, y). Let σ_2 be a winning strategy for player I at y.

We proceed to define a strategy τ_1 for player II at x. Let $\tau_1(\emptyset, X) = x_1$, where x_1 is such that $\tau(\emptyset, X \times Y) = (x_1, y_1)$. Define $\tau_1(\langle x_1 \rangle, U) = x_2$ where x_2 is such that

$$\tau(\langle (x_1, y_1) \rangle, U \times \sigma_2(\langle y_1 \rangle)) = (x_2, y_2).$$

(Of course, (x_2, y_2) depends on U. For the sake of simplicity, we do not indicate this in the notation.) If (x_i, y_i) has been defined for $i = 1, ..., n$, define $\tau_1(\langle x_1, ..., x_n \rangle, U) = x_{n+1}$, where x_{n+1} is such that

$$\tau(\langle (x_1, y_1), ..., (x_n, y_n) \rangle, U \times V) = (x_{n+1}, y_{n+1}),$$

where

$$V = \bigcap \{\sigma_2(F) \mid F \text{ is a subsequence of } \langle y_1, ..., y_n \rangle\} .$$

Now let σ_1 be a counterstrategy for τ_1. Define $\sigma : \mathcal{S}(X \times Y) \to \mathcal{T}_{(x,y)}(X \times Y)$ by

$$\sigma(\langle (x_1, y_1), ..., (x_n, y_n) \rangle) = (\sigma_1(\langle x_1, ..., x_n \rangle) \times \sigma_2(\langle y_1, ..., y_n \rangle)).$$

Then if $\langle (x_1, y_1), (x_2, y_2), ... \rangle$ is the result of the game determined by σ

and τ, clearly $\langle x_1, x_2, ...\rangle$ is the result of the game determined by σ_1 and τ_1, and $\langle y_1, y_2, ...\rangle$ is a σ_2-sequence. Then $(x_n, y_n) \to (x, y)$, and so σ is a counterstrategy for τ. Thus $X \times Y$ is a w-space. □

A Σ-*subspace* of a product space $X = \Pi\{X_\alpha \mid \alpha \in A\}$ is a subspace consisting of all points of X which differ from a given point $x \in X$ at no more than countably many coordinates. Noble [8] has proved that any Σ-subspace of the product of first countable spaces is a Fréchet space. The following theorem is a significant generalization of this. We remark that Arhangel'skii [1] has generalized Noble's result in a different direction by proving that a Σ-product of bi-sequential spaces is countably bi-sequential.

Theorem 4.6. *Any Σ-subspace of the product of* W-*space is a* W-*space.*
(Note that this theorem also generalizes Theorem 4.1.)

Proof. Let $X = \Pi\{X_\alpha \mid \alpha \in A\}$, where each X_α is a W-space. Suppose X' is a Σ-subspace of X. Pick $x = (x_\alpha)_{\alpha\in A} \in X'$, and let σ_α be a winning strategy for player I at $x_\alpha \in X_\alpha$. If U_α is open in X_α, let U_α^* denote the subbasic open set $\{y \in X' \mid y_\alpha \in U_\alpha\}$. For each $y \in X'$, choose an enumeration $\{\alpha_n(y) \mid n = 1, 2, ...\}$ of the coordinates at which y differs from x.

We proceed to define a strategy σ for player I at x. If $y(1)$ is the first choice of player II, let

$$\sigma(\langle y(1)\rangle) = \sigma_\alpha(y(1)_\alpha)^*,$$

where $\alpha = \alpha_1(y(1))$. If $y(2)$ is player II's next choice, let

$$\sigma(\langle y(1), y(2)\rangle) =$$

$$\bigcap \{\sigma_\alpha(F)^* \mid F \text{ is a subsequence of } \langle y(1)_\alpha, y(2)_\alpha\rangle, \alpha = \alpha_i(y(j)), i \leqslant 2, j \leqslant 2\}.$$

In general, define

$$\sigma(\langle y(1), ..., y(n)\rangle) =$$

$$\bigcap\{\sigma_\alpha(F)^* \mid F \text{ is a subsequence of } \langle y(1)_\alpha, ..., y(n)_\alpha\rangle, \alpha = \alpha_i(y(j)), i \leqslant n, j \leqslant n\}.$$

Suppose $\langle y(1), y(2), \ldots \rangle$ is a σ-sequence. We need to show that $y(n)_\alpha \to x_\alpha$ for each $\alpha \in A$. If $\alpha \neq \alpha_i(y(j))$ for some $(i, j) \in \mathbf{N} \times \mathbf{N}$, then $y(n)_\alpha = x_\alpha$, so suppose $\alpha = \alpha_k(y(m))$. Then if $n_0 = \max\{k, m\}$ and $n \geq n_0$, we have $y(n+1)_\alpha \in \sigma_\alpha(\langle y(n_0)_\alpha, \ldots, y(n)_\alpha \rangle)$. Thus $y(n)_\alpha \to x_\alpha$, and the proof is finished. □

5. Examples

As mentioned earlier, the one-point compactification of a discrete space is always a W-space (simply force player II to choose a different point each time). Thus the one-point compactification of a discrete space of measurable cardinality yields a W-space which is not bi-sequential [7, Example 10.15]. However, there are examples which do not depend on the existence of measurable cardinals.

Example 5.1. *A W-space X which is not a bi-sequential space.*

Proof. Let A be some uncountable set, and let $Y = \Pi_{\alpha \in A} \{0, 1\}_\alpha$. Let A be the subspace of Y consisting of all points $y \in Y$ such that $y_\alpha \neq 0$ for at most countably many α. By Theorem 4.6, X is a W-space. Arhangel'skii [1] has observed that X is not bi-sequential. □

Example 5.2. *A compact first-countable space X, and a perfect mapping $f: X \to Y$ such that Y, which must be bi-sequential, is not a W-space.*

Proof. In [7], Michael constructs a class of bi-sequential compact Hausdorff spaces which are not first-countable. We follow his construction, letting Z be the top and bottom edges of the square with the lexicographic order. Let $X = Z \times Z$. Since Z is not metrizable, the diagonal Δ of $Z \times Z$ is not G_δ. Let $Y = X/\Delta$, and let $f: X \to Y$ be the quotient map. Y is separable but not first-countable, thus Y is not a W-space. □

Consider again the c-spaces in which every countable subset is first-countable. This class seems to be very close to the W-spaces. However, we have not been able to prove that the product of two such spaces is again in the class. Of course, in the product we still have that every countable subset is first-countable. The problem is, perhaps it is not a c-space.

Example 5.3. *Two Fréchet spaces X and Y such that $X \times Y$ is not a c-space.*
(A similar example, without proof, was mentioned by Arhangel'skiĭ in [1].)

Proof. Let $Z = I \times I$. For $x \in I$, define

$$1_x = \{(x, y) \mid y \in I\}, \qquad 1^x = \{(y, x) \mid y \in I\}.$$

Let X be the set Z with $\{(x, 0) \mid x \in I\}$ identified to a point p_0, and endowed with the following topology: all points except p_0 are discrete, and a set U containing p_0 is open if and only if $U \cap 1_x$ is cofinite in 1_x for every $x \in I$. That is, X is the disjoint union of a continuum of one-point compactifications of a discrete space of cardinality continuum, with the non-discrete points identified to a single point. Thus X is a Fréchet space. Define Y similarly, with $\{(0, y) \mid y \in I\}$ identified to a point q_0, and $U \cap 1^y$ cofinite in 1^y for every open set U containing q_0 and $y \in I$.

Let

$$D = \{(x, y) \in X \times Y \mid x \neq p_0, y \neq q_0, \text{ and } x = y\},$$

where $x = y$ means that x and y are the same point in $I \times I$. We claim $(p_0, q_0) \in \overline{D}$. To see this, pick a countable subset $\{x_n \mid n \in \mathbf{N}\} \subset I$. If $(p_0, q_0) \in U \times V$ open in $X \times Y$, there exists $y \in I$, $y \neq 0$, such that $(x_n, y) \in U$ for all $n \in \mathbf{N}$ (since for each n there are only finitely many $y \in I$ such that $(x_n, y) \notin U$). Since $1^y \cap V$ is cofinite in 1^y, $(x_n, y) \in V$ for some $n \in \mathbf{N}$. Thus $(U \times V) \cap D \neq \emptyset$, so $(p_0, q_0) \in \overline{D}$.

It remains to prove that (p_0, q_0) is not in the closure of any countable subset of D. Consider

$$W = \{(1/m, 1/n) \in I \times I \mid m, n \in \mathbf{N}\}.$$

It is fairly easy to see (though messy to prove rigorously) that every countable subset of $D \cup \{(p_0, q_0)\}$ is homeomorphic to a subspace of $[D \cap (W \times W)] \cup \{(p_0, q_0)\}$. However, (p_0, q_0) is not in the closure of $D \cap (W \times W)$. To see this, let

$$U = \{p_0\} \cup \{(1/n, 1/m) \mid n < m\},$$

$$V = \{q_0\} \cup \{(1/n, 1/m) \mid n > m\}.$$

$U \times V$ is an open set containing (p_0, q_0) which does not intersect $D \cap (W \times W)$. This finishes the proof. □

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